
Click Here

Anatomy of CES Production/Utility Functions in Three Dimensions Peter Fuleky Department of Economics, University of Washington October 2006 Decreasing returns to scale: r=0.5 (Strongly concave production/utility function) Increasing returns to scale: r=1.5 (Quasiconcave production/utility function) Constant returns to scale: r=1 (Weakly concave production/utility function) Constant returns to scale with asymmetric weighting: r=1, lambda=0.8 (Weakly concave production/utility function) Constant returns to scale with asymmetric productivity/utility: r=1, T=2 (Weakly concave production/utility function) Literature and further reading: The Structure of Economics, 3rd ed., Eugene Silberberg MATLAB Documentation, MathWorks Anatomy of Cobb-Douglas Functions in 3D Printable version of this document (.pdf format) Concept in economicsConstant elasticity of substitution (CES) is a common specification of many production functions and utility functions in neoclassical economics. CES holds that the ability to substitute one input factor with another (for example labour with capital) to maintain the same level of production stays constant over different production levels. For utility functions, CES means the consumer has constant preferences of how they would like to substitute different goods (for example labour with consumption) while keeping the same level of utility, for all levels of utility. What this means is that both producers and consumers have similar input structures and preferences no matter the level of output or utility. The vital economic element of the measure is that it provided the producer a clear picture of how to move between different modes or types of production, for example between modes of production relying on more labour. Several economists have featured in the topic and have contributed in the final finding of the constant. They include Tom McKenzie, John Hicks and Joan Robinson. Specifically, it arises in a particular type of aggregator function which combines two or more types of consumption goods, or two or more types of production inputs into an aggregate quantity. This aggregator function exhibits constant elasticity of substitution.Despite having several factors of production in substitutability, the most common are the forms of elasticity of substitution. On the contrary of restricting direct empirical evaluation, the constant Elasticity of Substitution are simple to use and hence are widely used.[1] McFadden states that; The constant E.S assumption is a restriction on the form of production possibilities, and one can characterize the class of production functions which have this property. This has been done by Arrow-Chenery-Minhas-Solow for the two-factor production case.[1] The CES production function is a neoclassical production function that displays constant elasticity of substitution. In other words, the production technology has a constant percentage change in factor (e.g. labour and capital) proportions due to a percentage change in marginal rate of technical substitution. The two factor (capital, labor) CES production function introduced by Solow,[2] and later made popular by Arrow, Chenery, Minhas, and Solow is:[3][4][5][6] Q = F (a K + (1 a) L) {\displaystyle Q=F\cdot \left(a\cdot K^{\rho }+(1-a)\cdot L^{\rho }\right)^{\frac {\upsilon }{\rho }}} where Q {\displaystyle Q} = Quantity of output F {\displaystyle F} = Total Factor Productivity a {\displaystyle a} = Share parameter K {\displaystyle K} , L {\displaystyle L} = Quantities of primary production factors (Capital and Labor) {\displaystyle \rho } = 1 {\displaystyle {\frac {\sigma -1}{\sigma }}} = Substitution parameter {\displaystyle \sigma } = 1 1 {\displaystyle {\frac {1}{1-\rho }}} = Elasticity of substitution {\displaystyle \upsilon } = degree of homogeneity of the production function. Where {\displaystyle \upsilon } = 1 (Constant return to scale), {\displaystyle \upsilon } < 1 (Decreasing return to scale), {\displaystyle \upsilon } > 1 (Increasing return to scale).As its name suggests, the CES production function exhibits constant elasticity of substitution between capital and labor. Leontief, linear and CobbDouglas functions are special cases of the CES production function. That is, If {\displaystyle \rho } approaches 1, we have a linear or perfect substitutes function;If {\displaystyle \rho } approaches zero in the limit, we get the CobbDouglas production function;If {\displaystyle \rho } approaches negative infinity we get the Leontief or perfect complements production function.The general form of the CES production function, with n inputs, is:[7] Q = F [i = 1 n a i X i r] 1 r {\displaystyle Q=F\cdot \left[\sum _{i=1}^{n}a_{i}X_{i}^{r}\ \right]^{\frac {1}{r}}} where Q {\displaystyle Q} = Quantity of output F {\displaystyle F} = Total Factor Productivity a i {\displaystyle a_{i}} = Share parameter of input i, i = 1 n a i = 1 {\displaystyle \sum _{i=1}^{n}a_{i}=1} X i {\displaystyle X_{i}} = Quantities of factors of production (i = 1,2...n) s = 1 1 r {\displaystyle s={\frac {1}{1-r}}} = Elasticity of substitution.Extending the CES (Solow) functional form to accommodate multiple factors of production creates some problems. However, there is no completely general way to do this. Uzawa showed the only possible n-factor production functions (n>2) with constant partial elasticities of substitution require either that all elasticities between pairs of factors be identical, or if any differ, these all must equal each other and all remaining elasticities must be unity.[8] This is true for any production function. This means the use of the CES functional form for more than 2 factors will generally mean that there is not constant elasticity of substitution among all factors.Nested CES functions are commonly found in partial equilibrium and general equilibrium models. Different nests (levels) allow for the introduction of the appropriate elasticity of substitution.The same CES functional form arises as a utility function in consumer theory. For example, if there exist n {\displaystyle n} types of consumption goods x i {\displaystyle x_{i}} , then aggregate consumption X {\displaystyle X} could be defined using the CES aggregator: X = [i = 1 n a i 1 s x i s 1 s] s s 1 . {\displaystyle X=\left[\sum _{i=1}^{n}a_{i}^{\frac {1}{s}}x_{i}^{\frac {s-1}{s}}\ \right]^{\frac {s}{s-1}}.} Here again, the coefficients a i {\displaystyle a_{i}} are share parameters, and s {\displaystyle s} is the elasticity of substitution. Therefore, the consumption goods x i {\displaystyle x_{i}} are perfect substitutes when s {\displaystyle s} approaches infinity and perfect complements when s {\displaystyle s} approaches zero. In the case where s {\displaystyle s} approaches one is again a limiting case where L'Hpital's Rule applies. The CES aggregator is also sometimes called the Armington aggregator, which was discussed by Armington (1969).[9]CES utility functions are a special case of homothetic preferences.The following is an example of a CES utility function for two goods, x {\displaystyle x} and y {\displaystyle y} , with equal shares:[10]:112 u (x , y) = (x r + y r) 1 / r . {\displaystyle u(x,y)=(x^{r}+y^{r})^{1/r}.} The expenditure function in this case is: e (p x , p y , u) = (p x r / (r 1) + p y r / (r 1)) (r 1) / r u . {\displaystyle e(p_{x},p_{y},u)=(p_{x}^{r/(r-1)}+p_{y}^{r/(r-1)})^{(r-1)/r}\cdot u.} The indirect utility function is its inverse: v (p x , p y , I) = (p x r / (r 1) + p y r / (r 1)) (1 r) / r I . {\displaystyle v(p_{x},p_{y},I)=(p_{x}^{r/(r-1)}+p_{y}^{r/(r-1)})^{(1-r)/r}\cdot I.} The demand functions are: x (p x , p y , I) = p x 1 / (r 1) p x r / (r 1) + p y r / (r 1) I , {\displaystyle x(p_{x},p_{y},I)={\frac {p_{x}^{1/(r-1)}}{p_{x}^{r/(r-1)}+p_{y}^{r/(r-1)}}}\cdot I,} y (p x , p y , I) = p y 1 / (r 1) p x r / (r 1) + p y r / (r 1) I . {\displaystyle y(p_{x},p_{y},I)={\frac {p_{y}^{1/(r-1)}}{p_{x}^{r/(r-1)}+p_{y}^{r/(r-1)}}}\cdot I.} A CES utility function is one of the cases considered by Dixit and Stiglitz (1977) in their study of optimal product diversity in a context of monopolistic competition.[11]Note the difference between CES utility and isoelastic utility: the CES utility function is an ordinal utility function that represents preferences on sure consumption commodity bundles, while the isoelastic utility function is a cardinal utility function that represents preferences on lotteries. A CES indirect (dual) utility function has been used to derive utility-consistent brand demand systems where category demands are determined endogenously by a multi-category, CES indirect (dual) utility function. It has also been shown that CES preferences are self-dual and that both primal and dual CES preferences yield systems of indifference curves that may exhibit any degree of convexity.[12]^ a b McFadden, Daniel (June 1963). "Constant Elasticity of Substitution Production Functions". The Review of Economic Studies. 30 (2): 7383. doi:10.2307/2295804. ISSN0034-6527. JSTOR2295804.^ Solow, R.M (1956). "A contribution to the theory of economic growth". The Quarterly Journal of Economics. 70 (1): 6594. doi:10.2307/1884513. hdl:10338.dmlcz/143862. JSTOR1884513.^ Arrow, K. J.; Chenery, H. B.; Minhas, B. S.; Solow, R. M. (1961). "Capital-labor substitution and economic efficiency". Review of Economics and Statistics. 43 (3): 225250. doi:10.2307/1927286. JSTOR1927286.^ Jorgensen, Dale W. (2000). Econometrics, vol. 1: Econometric Modelling of Producer Behavior. Cambridge, MA: MIT Press. p.2. ISBN978-0-262-10082-3.^ Klump, R; McAdam, P; Willman, A. (2007). "Factor Substitution and Factor Augmenting Technical Progress in the US: A Normalized Supply-Side System Approach". Review of Economics and Statistics. 89 (1): 183192. doi:10.1162/rest.89.1.183. hdl:10419/152801. S2CID57570638.^ de La Grandville, Olivier (2016). Economic Growth: A Unified Approach. Cambridge University Press. doi:10.1017/9781316335703. ISBN9781316335703.^ tedb/Courses/GraduateTheoryUCSB/elasticity%20of%20substitutionrevised.tex.pdf Archived 2022-01-01 at the Wayback Machine [bare URL PDF]^ Uzawa, H (1962). "Production functions with constant elasticities of substitution". Review of Economic Studies. 29 (4): 291299. doi:10.2307/2296305. JSTOR2296305.^ Armington, P. S. (1969). "A theory of demand for products distinguished by place of production". IMF Staff Papers. 16 (1): 159178. JSTOR3866403.^ Varian, Hal (1992). Microeconomic Analysis (Thirded.). New York: Norton. ISBN0-393-95735-7.^ Dixit, Avinash; Stiglitz, Joseph (1977). "Monopolistic Competition and Optimum Product Diversity". American Economic Review. 67 (3): 297308. JSTOR1831401.^ Baltas, George (2001). "Utility-Consistent Brand Demand Systems with Endogenous Category Consumption: Principles and Marketing Applications". Decision Sciences. 32 (3): 399421. doi:10.1111/j.1540-5915.2001.tb00965.x.Anatomy of CES Type Production Functions in 3DClosed form solution for a firm with an N-dimensional CES technologyMonopolists revenue functionRetrieved from " in economicsConstant elasticity of substitution (CES) is a common specification of many production functions and utility functions in neoclassical economics. CES holds that the ability to substitute one input factor with another (for example labour with capital) to maintain the same level of production stays constant over different production levels. For utility functions, CES means the consumer has constant preferences of how they would like to substitute different goods (for example labour with consumption) while keeping the same level of utility, for all levels of utility. What this means is that both producers and consumers have similar input structures and preferences no matter the level of output or utility. The vital economic element of the measure is that it provided the producer a clear picture of how to move between different modes or types of production, for example between modes of production relying on more labour. Several economists have featured in the topic and have contributed in the final finding of the constant. They include Tom McKenzie, John Hicks and Joan Robinson. Specifically, it arises in a particular type of aggregator function which combines two or more types of consumption goods, or two or more types of production inputs into an aggregate quantity. This aggregator function exhibits constant elasticity of substitution.Despite having several factors of production in substitutability, the most common are the forms of elasticity of substitution. On the contrary of restricting direct empirical evaluation, the constant Elasticity of Substitution are simple to use and hence are widely used.[1] McFadden states that; The constant E.S assumption is a restriction on the form of production possibilities, and one can characterize the class of production functions which have this property. This has been done by Arrow-Chenery-Minhas-Solow for the two-factor production case.[1] The CES production function is a neoclassical production function that displays constant elasticity of substitution. In other words, the production technology has a constant percentage change in factor (e.g. labour and capital) proportions due to a percentage change in marginal rate of technical substitution. The two factor (capital, labor) CES production function introduced by Solow,[2] and later made popular by Arrow, Chenery, Minhas, and Solow is:[3][4][5][6] Q = F (a K + (1 a) L) {\displaystyle Q=F\cdot \left(a\cdot K^{\rho }+(1-a)\cdot L^{\rho }\right)^{\frac {\upsilon }{\rho }}} where Q {\displaystyle Q} = Quantity of output F {\displaystyle F} = Total Factor Productivity a {\displaystyle a} = Share parameter K {\displaystyle K} , L {\displaystyle L} = Quantities of primary production factors (Capital and Labor) {\displaystyle \rho } = 1 {\displaystyle {\frac {\sigma -1}{\sigma }}} = Substitution parameter {\displaystyle \sigma } = 1 1 {\displaystyle {\frac {1}{1-\rho }}} = Elasticity of substitution {\displaystyle \upsilon } = degree of homogeneity of the production function. Where {\displaystyle \upsilon } = 1 (Constant return to scale), {\displaystyle \upsilon } < 1 (Decreasing return to scale), {\displaystyle \upsilon } > 1 (Increasing return to scale).As its name suggests, the CES production function exhibits constant elasticity of substitution between capital and labor. Leontief, linear and CobbDouglas functions are special cases of the CES production function. That is, If {\displaystyle \rho } approaches 1, we have a linear or perfect substitutes function;If {\displaystyle \rho } approaches zero in the limit, we get the CobbDouglas production function;If {\displaystyle \rho } approaches negative infinity we get the Leontief or perfect complements production function.The general form of the CES production function, with n inputs, is:[7] Q = F [i = 1 n a i X i r] 1 r {\displaystyle Q=F\cdot \left[\sum _{i=1}^{n}a_{i}X_{i}^{r}\ \right]^{\frac {1}{r}}} where Q {\displaystyle Q} = Quantity of output F {\displaystyle F} = Total Factor Productivity a i {\displaystyle a_{i}} = Share parameter of input i, i = 1 n a i = 1 {\displaystyle \sum _{i=1}^{n}a_{i}=1} X i {\displaystyle X_{i}} = Quantities of factors of production (i = 1,2...n) s = 1 1 r {\displaystyle s={\frac {1}{1-r}}} = Elasticity of substitution.Extending the CES (Solow) functional form to accommodate multiple factors of production creates some problems. However, there is no completely general way to do this. Uzawa showed the only possible n-factor production functions (n>2) with constant partial elasticities of substitution require either that all elasticities between pairs of factors be identical, or if any differ, these all must equal each other and all remaining elasticities must be unity.[8] This is true for any production function. This means the use of the CES functional form for more than 2 factors will generally mean that there is not constant elasticity of substitution among all factors.Nested CES functions are commonly found in partial equilibrium and general equilibrium models. Different nests (levels) allow for the introduction of the appropriate elasticity of substitution.The same CES functional form arises as a utility function in consumer theory. For example, if there exist n {\displaystyle n} types of consumption goods x i {\displaystyle x_{i}} , then aggregate consumption X {\displaystyle X} could be defined using the CES aggregator: X = [i = 1 n a i 1 s x i s 1 s] s s 1 . {\displaystyle X=\left[\sum _{i=1}^{n}a_{i}^{\frac {1}{s}}x_{i}^{\frac {s-1}{s}}\ \right]^{\frac {s}{s-1}}.} Here again, the coefficients a i {\displaystyle a_{i}} are share parameters, and s {\displaystyle s} is the elasticity of substitution. Therefore, the consumption goods x i {\displaystyle x_{i}} are perfect substitutes when s {\displaystyle s} approaches infinity and perfect complements when s {\displaystyle s} approaches zero. In the case where s {\displaystyle s} approaches one is again a limiting case where L'Hpital's Rule applies. The CES aggregator is also sometimes called the Armington aggregator, which was discussed by Armington (1969).[9]CES utility functions are a special case of homothetic preferences.The following is an example of a CES utility function for two goods, x {\displaystyle x} and y {\displaystyle y} , with equal shares:[10]:112 u (x , y) = (x r + y r) 1 / r . {\displaystyle u(x,y)=(x^{r}+y^{r})^{1/r}.} The expenditure function in this case is: e (p x , p y , u) = (p x r / (r 1) + p y r / (r 1)) (r 1) / r u . {\displaystyle e(p_{x},p_{y},u)=(p_{x}^{r/(r-1)}+p_{y}^{r/(r-1)})^{(r-1)/r}\cdot u.} The indirect utility function is its inverse: v (p x , p y , I) = (p x r / (r 1) + p y r / (r 1)) (1 r) / r I . {\displaystyle v(p_{x},p_{y},I)=(p_{x}^{r/(r-1)}+p_{y}^{r/(r-1)})^{(1-r)/r}\cdot I.} The demand functions are: x (p x , p y , I) = p x 1 / (r 1) p x r / (r 1) + p y r / (r 1) I , {\displaystyle x(p_{x},p_{y},I)={\frac {p_{x}^{1/(r-1)}}{p_{x}^{r/(r-1)}+p_{y}^{r/(r-1)}}}\cdot I,} y (p x , p y , I) = p y 1 / (r 1) p x r / (r 1) + p y r / (r 1) I . {\displaystyle y(p_{x},p_{y},I)={\frac {p_{y}^{1/(r-1)}}{p_{x}^{r/(r-1)}+p_{y}^{r/(r-1)}}}\cdot I.} A CES utility function is one of the cases considered by Dixit and Stiglitz (1977) in their study of optimal product diversity in a context of monopolistic competition.[11]Note the difference between CES utility and isoelastic utility: the CES utility function is an ordinal utility function that represents preferences on sure consumption commodity bundles, while the isoelastic utility function is a cardinal utility function that represents preferences on lotteries. A CES indirect (dual) utility function has been used to derive utility-consistent brand demand systems where category demands are determined endogenously by a multi-category, CES indirect (dual) utility function. It has also been shown that CES preferences are self-dual and that both primal and dual CES preferences yield systems of indifference curves that may exhibit any degree of convexity.[12]^ a b McFadden, Daniel (June 1963). "Constant Elasticity of Substitution Production Functions". The Review of Economic Studies. 30 (2): 7383. doi:10.2307/2295804. ISSN0034-6527. JSTOR2295804.^ Solow, R.M (1956). "A contribution to the theory of economic growth". The Quarterly Journal of Economics. 70 (1): 6594. doi:10.2307/1884513. hdl:10338.dmlcz/143862. JSTOR1884513.^ Arrow, K. J.; Chenery, H. B.; Minhas, B. S.; Solow, R. M. (1961). "Capital-labor substitution and economic efficiency". Review of Economics and Statistics. 43 (3): 225250. doi:10.2307/1927286. JSTOR1927286.^ Jorgensen, Dale W. (2000). Econometrics, vol. 1: Econometric Modelling of Producer Behavior. Cambridge, MA: MIT Press. p.2. ISBN978-0-262-10082-3.^ Klump, R; McAdam, P; Willman, A. (2007). "Factor Substitution and Factor Augmenting Technical Progress in the US: A Normalized Supply-Side System Approach". Review of Economics and Statistics. 89 (1): 183192. doi:10.1162/rest.89.1.183. hdl:10419/152801. S2CID57570638.^ de La Grandville, Olivier (2016). Economic Growth: A Unified Approach. Cambridge University Press. doi:10.1017/9781316335703. ISBN9781316335703.^ tedb/Courses/GraduateTheoryUCSB/elasticity%20of%20substitutionrevised.tex.pdf Archived 2022-01-01 at the Wayback Machine [bare URL PDF]^ Uzawa, H (1962). "Production functions with constant elasticities of substitution". Review of Economic Studies. 29 (4): 291299. doi:10.2307/2296305. JSTOR2296305.^ Armington, P. S. (1969). "A theory of demand for products distinguished by place of production". IMF Staff Papers. 16 (1): 159178. JSTOR3866403.^ Varian, Hal (1992). Microeconomic Analysis (Thirded.). New York: Norton. ISBN0-393-95735-7.^ Dixit, Avinash; Stiglitz, Joseph (1977). "Monopolistic Competition and Optimum Product Diversity". American Economic Review. 67 (3): 297308. JSTOR1831401.^ Baltas, George (2001). "Utility-Consistent Brand Demand Systems with Endogenous Category Consumption: Principles and Marketing Applications". Decision Sciences. 32 (3): 399421. doi:10.1111/j.1540-5915.2001.tb00965.x.Anatomy of CES Type Production Functions in 3DClosed form solution for a firm with an N-dimensional CES technologyMonopolists revenue functionRetrieved from " Concept in economicsConstant elasticity of substitution (CES) is a common specification of many production functions and utility functions in neoclassical economics. CES holds that the ability to substitute one input factor with another (for example labour with capital) to maintain the same level of production stays constant over different production levels. For utility functions, CES means the consumer has constant preferences of how they would like to substitute different goods (for example labour with consumption) while keeping the same level of utility, for all levels of utility. What this means is that both producers and consumers have similar input structures and preferences no matter the level of output or utility. The vital economic element of the measure is that it provided the producer a clear picture of how to move between different modes or types of production, for example between modes of production relying on more labour. Several economists have featured in the topic and have contributed in the final finding of the constant. They include Tom McKenzie, John Hicks and Joan Robinson. Specifically, it arises in a particular type of aggregator function which combines two or more types of consumption goods, or two or more types of production inputs into an aggregate quantity. This aggregator function exhibits constant elasticity of substitution.Despite having several factors of production in substitutability, the most common are the forms of elasticity of substitution. On the contrary of restricting direct empirical evaluation, the constant Elasticity of Substitution are simple to use and hence are widely used.[1] McFadden states that; The constant E.S assumption is a restriction on the form of production possibilities, and one can characterize the class of production functions which have this property. This has been done by Arrow-Chenery-Minhas-Solow for the two-factor production case.[1] The CES production function is a neoclassical production function that displays constant elasticity of substitution. In other words, the production technology has a constant percentage change in factor (e.g. labour and capital) proportions due to a percentage change in marginal rate of technical substitution. The two factor (capital, labor) CES production function introduced by Solow,[2] and later made popular by Arrow, Chenery, Minhas, and Solow is:[3][4][5][6] Q = F (a K + (1 a) L) {\displaystyle Q=F\cdot \left(a\cdot K^{\rho }+(1-a)\cdot L^{\rho }\right)^{\frac {\upsilon }{\rho }}} where Q {\displaystyle Q} = Quantity of output F {\displaystyle F} = Total Factor Productivity a {\displaystyle a} = Share parameter K {\displaystyle K} , L {\displaystyle L} = Quantities of primary production factors (Capital and Labor) {\displaystyle \rho } = 1 {\displaystyle {\frac {\sigma -1}{\sigma }}} = Substitution parameter {\displaystyle \sigma } = 1 1 {\displaystyle {\frac {1}{1-\rho }}} = Elasticity of substitution {\displaystyle \upsilon } = degree of homogeneity of the production function. Where {\displaystyle \upsilon } = 1 (Constant return to scale), {\displaystyle \upsilon } < 1 (Decreasing return to scale), {\displaystyle \upsilon } > 1 (Increasing return to scale).As its name suggests, the CES production function exhibits constant elasticity of substitution between capital and labor. Leontief, linear and CobbDouglas functions are special cases of the CES production function. That is, If {\displaystyle \rho } approaches 1, we have a linear or perfect substitutes function;If {\displaystyle \rho } approaches zero in the limit, we get the CobbDouglas production function;If {\displaystyle \rho } approaches negative infinity we get the Leontief or perfect complements production function.The general form of the CES production function, with n inputs, is:[7] Q = F [i = 1 n a i X i r] 1 r {\displaystyle Q=F\cdot \left[\sum _{i=1}^{n}a_{i}X_{i}^{r}\ \right]^{\frac {1}{r}}} where Q {\displaystyle Q} = Quantity of output F {\displaystyle F} = Total Factor Productivity a i {\displaystyle a_{i}} = Share parameter of input i, i = 1 n a i = 1 {\displaystyle \sum _{i=1}^{n}a_{i}=1} X i {\displaystyle X_{i}} = Quantities of factors of production (i = 1,2...n) s = 1 1 r {\displaystyle s={\frac {1}{1-r}}} = Elasticity of substitution.Extending the CES (Solow) functional form to accommodate multiple factors of production creates some problems. However, there is no completely general way to do this. Uzawa showed the only possible n-factor production functions (n>2) with constant partial elasticities of substitution require either that all elasticities between pairs of factors be identical, or if any differ, these all must equal each other and all remaining elasticities must be unity.[8] This is true for any production function. This means the use of the CES functional form for more than 2 factors will generally mean that there is not constant elasticity of substitution among all factors.Nested CES functions are commonly found in partial equilibrium and general equilibrium models. Different nests (levels) allow for the introduction of the appropriate elasticity of substitution.The same CES functional form arises as a utility function in consumer theory. For example, if there exist n {\displaystyle n} types of consumption goods x i {\displaystyle x_{i}} , then aggregate consumption X {\displaystyle X} could be defined using the CES aggregator: X = [i = 1 n a i 1 s x i s 1 s] s s 1 . {\displaystyle X=\left[\sum _{i=1}^{n}a_{i}^{\frac {1}{s}}x_{i}^{\frac {s-1}{s}}\ \right]^{\frac {s}{s-1}}.} Here again, the coefficients a i {\displaystyle a_{i}} are share parameters, and s {\displaystyle s} is the elasticity of substitution. Therefore, the consumption goods x i {\displaystyle x_{i}} are perfect substitutes when s {\displaystyle s} approaches infinity and perfect complements when s {\displaystyle s} approaches zero. In the case where s {\displaystyle s} approaches one is again a limiting case where L'Hpital's Rule applies. The CES aggregator is also sometimes called the Armington aggregator, which was discussed by Armington (1969).[9]CES utility functions are a special case of homothetic preferences.The following is an example of a CES utility function for two goods, x {\displaystyle x} and y {\displaystyle y} , with equal shares:[10]:112 u (x , y) = (x r + y r) 1 / r . {\displaystyle u(x,y)=(x^{r}+y^{r})^{1/r}.} The expenditure function in this case is: e (p x , p y , u) = (p x r / (r 1) + p y r / (r 1)) (r 1) / r u . {\displaystyle e(p_{x},p_{y},u)=(p_{x}^{r/(r-1)}+p_{y}^{r/(r-1)})^{(r-1)/r}\cdot u.} The indirect utility function is its inverse: v (p x , p y , I) = (p x r / (r 1) + p y r / (r 1)) (1 r) / r I . {\displaystyle v(p_{x},p_{y},I)=(p_{x}^{r/(r-1)}+p_{y}^{r/(r-1)})^{(1-r)/r}\cdot I.} The demand functions are: x (p x , p y , I) = p x 1 / (r 1) p x r / (r 1) + p y r / (r 1) I , {\displaystyle x(p_{x},p_{y},I)={\frac {p_{x}^{1/(r-1)}}{p_{x}^{r/(r-1)}+p_{y}^{r/(r-1)}}}\cdot I,} y (p x , p y , I) = p y 1 / (r 1) p x r / (r 1) + p y r / (r 1) I . {\displaystyle y(p_{x},p_{y},I)={\frac {p_{y}^{1/(r-1)}}{p_{x}^{r/(r-1)}+p_{y}^{r/(r-1)}}}\cdot I.} A CES utility function is one of the cases considered by Dixit and Stiglitz (1977) in their study of optimal product diversity in a context of monopolistic competition.[11]Note the difference between CES utility and isoelastic utility: the CES utility function is an ordinal utility function that represents preferences on sure consumption commodity bundles, while the isoelastic utility function is a cardinal utility function that represents preferences on lotteries. A CES indirect (dual) utility function has been used to derive utility-consistent brand demand systems where category demands are determined endogenously by a multi-category, CES indirect (dual) utility function. It has also been shown that CES preferences are self-dual and that both primal and dual CES preferences yield systems of indifference curves that may exhibit any degree of convexity.[12]^ a b McFadden, Daniel (June 1963). "Constant Elasticity of Substitution Production Functions". The Review of Economic Studies. 30 (2): 7383. doi:10.2307/2295804. ISSN0034-6527. JSTOR2295804.^ Solow, R.M (1956). "A contribution to the theory of economic growth". The Quarterly Journal of Economics. 70 (1): 6594. doi:10.2307/1884513. hdl:10338.dmlcz/143862. JSTOR1884513.^ Arrow, K. J.; Chenery, H. B.; Minhas, B. S.; Solow, R. M. (1961). "Capital-labor substitution and economic efficiency". Review of Economics and Statistics. 43 (3): 225250. doi:10.2307/1927286. JSTOR1927286.^ Jorgensen, Dale W. (2000). Econometrics, vol. 1: Econometric Modelling of Producer Behavior. Cambridge, MA: MIT Press. p.2. ISBN978-0-262-10082-3.^ Klump, R; McAdam, P; Willman, A. (2007). "Factor Substitution and Factor Augmenting Technical Progress in the US: A Normalized Supply-Side System Approach". Review of Economics and Statistics. 89 (1): 183192. doi:10.1162/rest.89.1.183. hdl:10419/152801. S2CID57570638.^ de La Grandville, Olivier (2016). Economic Growth: A Unified Approach. Cambridge University Press. doi:10.1017/9781316335703. ISBN9781316335703.^ tedb/Courses/GraduateTheoryUCSB/elasticity%20of%20substitutionrevised.tex.pdf Archived 2022-01-01 at the Wayback Machine [bare URL PDF]^ Uzawa, H (1962). "Production functions with constant elasticities of substitution". Review of Economic Studies. 29 (4): 291299. doi:10.2307/2296305. JSTOR2296305.^ Armington, P. S. (1969). "A theory of demand for products distinguished by place of production". IMF Staff Papers. 16 (1): 159178. JSTOR3866403.^ Varian, Hal (1992). Microeconomic Analysis (Thirded.). New York: Norton. ISBN0-393-95735-7.^ Dixit, Avinash; Stiglitz, Joseph (1977). "Monopolistic Competition and Optimum Product Diversity". American Economic Review. 67 (3): 297308. JSTOR1831401.^ Baltas, George (2001). "Utility-Consistent Brand Demand Systems with Endogenous Category Consumption: Principles and Marketing Applications". Decision Sciences. 32 (3): 399421. doi:10.1111/j.1540-5915.2001.tb00965.x.Anatomy of CES Type Production Functions in 3DClosed form solution for a firm with an N-dimensional CES technologyMonopolists revenue functionRetrieved from " Teaching Notes Research Notes In this post, Ill carefully explain the derivation of cost function from a CES production function, as well as the derivation of translog (transcendental logarithmic) production and cost functions.flowchart TB subgraph Production A[Production Function] -. approximation .-> D(Translog Production Function) end subgraph Cost B[Cost Function] -. approximation .-> C(Translog Cost Function) end A == Conversion via Duality ==> BFigure1: A diagram of the relationship between production functions and cost functions.Before I start, the graph above illustrate the relations. Specifically, we can derive the cost function from a CES production function via the duality theorem. Translog production and translog cost functions are approximations to the production and corresponding cost function, respectively, via Taylor expansion. Lets start from the a general production function, CES (Constant Elasticity of Substitution).The standard CES production function with two factors \(X_1\) and \(X_2\) is given by:\[\begin{equation}\label{eq:ces-production}Q = A \left(\alpha_1 X_1^{\rho} + \alpha_2 X_2^{\rho} \right)^{\frac{1}{\rho}}\end{equation}\tag{1}\]where \(\alpha_1+\alpha_2=1\), \(A\) is a scale parameter, \(\alpha\) is the distribution parameter, and \(\rho\) is the substitution parameter.The Cobb-Douglas production function is a special case of the CES function when \(\rho \to 0\):\[Q = A X_1^{\alpha} X_2^{1-\alpha}\] Taking the natural logarithm of both sides of Equation Equation1, we get:\[\begin{equation}\label{eq:log-form-ces-production}\ln Q = \ln A + \frac{1}{\rho} \ln \left[\alpha X_1^{\rho} + (1-\alpha) X_2^{\rho} \right]\end{equation}\tag{2}\]The Taylor expansion of \(\frac{1}{\rho} \ln \left[\alpha X_1^{\rho} + (1-\alpha) X_2^{\rho} \right]\) around \(\rho=0\) is 11This is computed in Mathematica:expr = Series[1/rho * Log[alpha*X1^rho + (1 - alpha)*X2^rho], {rho, 0, 1}];simplifiedExpr = FullSimplify[expr];TeXForm[simplifiedExpr]\[\begin{equation}\label{eq:taylor-expansion-of-ces}\alpha \ln X_1 + (1-\alpha) \ln X_2-\frac{1}{2} \rho \left[(\alpha -1) \alpha (\ln X_1-\ln X_2)^2\right]+O\left(\rho ^2\right)\end{equation}\tag{3}\]Omitting \(O\left(\rho ^2\right)\) and substituting the Taylor expansion into Equation Equation2, we have\[\begin{align}\label{eq:log-production-with-taylor}\ln Q &= \ln A \\&+ \alpha \ln X_1 + (1-\alpha) \ln X_2-\frac{1}{2} \rho \left[(\alpha -1) \alpha (\ln X_1-\ln X_2)^2\right] onumber\end{align}\tag{4}\]which clearly is a function of \(\ln X_1\), \(\ln X_2\) and their interaction terms.We can therefore reparameterize Equation Equation4 and get the Translog production function:\[\begin{align}\ln Q &= a_0 + a_1 \ln X_1 + a_2 \ln X_2 \\&+ b_{11} (\ln X_1)^2 + b_{22} (\ln X_2)^2 + b_{12} \ln X_1 \ln X_2 onumber\end{align}\]Here, \(a_1\) and \(a_2\) are coefficients that capture the first-order effects, and \(b_{11}\), \(b_{22}\), and \(b_{12}\) are coefficients that capture the second-order effects.If we use fist-order Taylor expansion in Equation Equation3 instead, we will end up with a log-linear production function. Given the CES production function Equation1, we can derive the cost function via the duality theorem.The production function describes the maximum output \(Q\) that can be produced given the input factors.Given a production function and input prices, the firm aims to minimize its costs subject to the constraint of producing a given output level \(Q\). This leads to a cost minimization problem.Cost minimization and the production maximization are essentially dual to each other. The conditions that solve one problem can be used to solve the other. This is a manifestation of the more general concept of duality in optimization theory.Recall that the CES production function is\[Q = A \left(\alpha_1 X_1^{\rho} + \alpha_2 X_2^{\rho} \right)^{\frac{1}{\rho}}\]The firms cost function is\[\begin{equation}\label{eq:cost-function}C = w_1 X_1 + w_2 X_2\end{equation}\tag{5}\]where \(w_1\) and \(w_2\) are the factor prices. To derive the cost function from the given CES production function, we need to find the minimum cost of producing a given level of output \(Q\) given input prices \(w_1\) and \(w_2\).The cost minimization problem is:\[\begin{equation}\min_{X_1, X_2} \quad C=w_1 X_1 + w_2 X_2\end{equation}\]subject to:\[\begin{equation}A \left(\alpha_1 X_1^{\rho} + \alpha_2 X_2^{\rho} \right)^{\frac{1}{\rho}} = Q\end{equation}\]This part is math-heavy. The derived cost function is given by Equation Equation11.The Lagrangian for this problem is:\[\begin{equation}\mathcal{L} = w_1 X_1 + w_2 X_2 + \lambda \left[Q - A \left(\alpha_1 X_1^{\rho} + \alpha_2 X_2^{\rho} \right)^{\frac{1}{\rho}} \right]\end{equation}\]Take the first-order conditions:\[\begin{align}\frac{\partial \mathcal{L}}{\partial X_1} &= w_1 - \lambda A \alpha_1 \rho X_1^{\rho-1} \left(\alpha_1 X_1^{\rho} + \alpha_2 X_2^{\rho} \right)^{\frac{1}{\rho}-1} = 0 \\\frac{\partial \mathcal{L}}{\partial X_2} &= w_2 - \lambda A \alpha_2 \rho X_2^{\rho-1} \left(\alpha_1 X_1^{\rho} + \alpha_2 X_2^{\rho} \right)^{\frac{1}{\rho}-1} = 0 \\\frac{\partial \mathcal{L}}{\partial \lambda} &= Q - A \left(\alpha_1 X_1^{\rho} + \alpha_2 X_2^{\rho} \right)^{\frac{1}{\rho}} = 0\end{align}\]Solve the first two equations for \(\lambda\):\[\begin{equation}\label{eq:lambda}\lambda = \frac{w_1}{A \alpha_1 \rho X_1^{\rho-1} \left(\alpha_1 X_1^{\rho} + \alpha_2 X_2^{\rho} \right)^{\frac{1}{\rho}-1}} = \frac{w_2}{A \alpha_2 \rho X_2^{\rho-1} \left(\alpha_1 X_1^{\rho} + \alpha_2 X_2^{\rho} \right)^{\frac{1}{\rho}-1}}\end{equation}\tag{6}\]Simplifying Equation6, we get:\[\begin{equation}\label{eq:lambda-equal}w_1 X_2^{\rho-1} \alpha_2 = w_2 X_1^{\rho-1} \alpha_1\end{equation}\tag{7}\]Manipulating Equation7, we have:\[\begin{equation}\frac{X_1}{X_2} = \left(\frac{\alpha_2 w_1}{\alpha_1 w_2}\right)^{\frac{1}{\rho-1}}\end{equation}\]so that\[\begin{align}(\alpha_2 w_1)^{\frac{\rho}{\rho-1}} X_2^{\rho} &= (\alpha_1 w_2)^{\frac{\rho}{\rho-1}} X_1^{\rho} \\\left(w_1^{\frac{\rho}{\rho-1}} \alpha_1^{\frac{-1}{\rho-1}}\right) \alpha_2 X_2^{\rho} &= \left(w_2^{\frac{\rho}{\rho-1}} \alpha_2^{\frac{-1}{\rho-1}}\right) \alpha_1 X_1^{\rho}\end{align}\]Adding \(\left(w_2^{\frac{\rho}{\rho-1}} \alpha_2^{\frac{-1}{\rho-1}}\right) \alpha_2 X_2^{\rho}\) to both sides, we have\[\begin{align}\left(w_1^{\frac{\rho}{\rho-1}} \alpha_1^{\frac{-1}{\rho-1}}\right) \alpha_2 X_2^{\rho} + \left(w_2^{\frac{\rho}{\rho-1}} \alpha_2^{\frac{-1}{\rho-1}}\right) \alpha_2 X_2^{\rho}&= \left(w_2^{\frac{\rho}{\rho-1}} \alpha_2^{\frac{-1}{\rho-1}}\right) \alpha_1 X_1^{\rho} + \left(w_2^{\frac{\rho}{\rho-1}} \alpha_2^{\frac{-1}{\rho-1}}\right) \alpha_2 X_2^{\rho} onumber \\\left(w_1^{\frac{\rho}{\rho-1}} \alpha_1^{\frac{-1}{\rho-1}} + w_2^{\frac{\rho}{\rho-1}} \alpha_2^{\frac{-1}{\rho-1}}\right) \alpha_2 X_2^{\rho}&= \left(w_2^{\frac{\rho}{\rho-1}} \alpha_2^{\frac{-1}{\rho-1}}\right) \left(\alpha_1 X_1^{\rho} + \alpha_2 X_2^{\rho}\right)\end{align}\]Raise both sides to the power of \(\frac{1}{\rho}\), we have\[\begin{equation}\label{eq:x2_before_simplification}\left(w_1^{\frac{\rho}{\rho-1}} \alpha_1^{\frac{-1}{\rho-1}} + w_2^{\frac{\rho}{\rho-1}} \alpha_2^{\frac{-1}{\rho-1}} \right)^{\frac{1}{\rho}} \alpha_2^{\frac{1}{\rho}} X_2= \left(w_2^{\frac{\rho}{\rho-1}} \alpha_2^{\frac{-1}{\rho-1}}\right)^{\frac{1}{\rho}} \left(\alpha_1 X_1^{\rho} + \alpha_2 X_2^{\rho}\right)^{\frac{1}{\rho}}\end{equation}\tag{8}\]Let \(K = \left(w_1^{\frac{\rho}{\rho-1}} \alpha_1^{\frac{-1}{\rho-1}} + w_2^{\frac{\rho}{\rho-1}} \alpha_2^{\frac{-1}{\rho-1}} \right)^{\frac{1}{\rho}}\), observe that \(\frac{Q}{A} = \left(\alpha_1 X_1^{\rho} + \alpha_2 X_2^{\rho}\right)^{\frac{1}{\rho}}\), we can simplify Equation8 to\[\begin{equation}K \alpha_2^{\frac{1}{\rho}} X_2 = \left(w_2^{\frac{\rho}{\rho-1}} \alpha_2^{\frac{-1}{\rho-1}}\right)^{\frac{1}{\rho}} \frac{Q}{A}\end{equation}\]Therefore, \(X_2\) is given by\[\begin{equation}\label{eq:x2}X_2 = K^{-1} w_2^{\frac{1}{\rho-1}} \alpha_2^{\frac{-1}{\rho-1}} \frac{Q}{A}\end{equation}\tag{9}\]We can similarly get \(X_1\)\[\begin{equation}\label{eq:x1}X_1 = K^{-1} w_1^{\frac{1}{\rho-1}} \alpha_1^{\frac{-1}{\rho-1}} \frac{Q}{A}\end{equation}\tag{10}\]Substituting Equation10 and Equation9 into the cost function Equation5, we have\[\begin{align}C &= w_1 X_1 + w_2 X_2 onumber \\&= K^{-1} w_1^{\frac{\rho}{\rho-1}} \alpha_1^{\frac{-1}{\rho-1}} \frac{Q}{A} + K^{-1} w_2^{\frac{\rho}{\rho-1}} \alpha_2^{\frac{-1}{\rho-1}} \frac{Q}{A} \\&= \frac{Q}{A} K^{-1} \left(w_1^{\frac{\rho}{\rho-1}} \alpha_1^{\frac{-1}{\rho-1}} + w_2^{\frac{\rho}{\rho-1}} \alpha_2^{\frac{-1}{\rho-1}}\right)\end{align}\]Since \(K = \left(w_1^{\frac{\rho}{\rho-1}} \alpha_1^{\frac{-1}{\rho-1}} + w_2^{\frac{\rho}{\rho-1}} \alpha_2^{\frac{-1}{\rho-1}} \right)^{\frac{1}{\rho}}\), we have the derived cost function:Cost function derived from CES production function\[\begin{equation}\label{eq:derived-cost-function}C = \frac{Q}{A} \left(w_1^{\frac{\rho}{\rho-1}} \alpha_1^{\frac{-1}{\rho-1}} + w_2^{\frac{\rho}{\rho-1}} \alpha_2^{\frac{-1}{\rho-1}}\right)^{\frac{\rho-1}{\rho}}\end{equation}\tag{11}\] Taking the natural logarithm of both sides of Equation Equation11, we get:\[\begin{equation}\label{eq:log-cost-function}\ln(C) = \ln \left(\frac{Q}{A} \right) + \frac{\rho-1}{\rho} \ln \left(w_1^{\frac{\rho}{\rho-1}} \alpha_1^{\frac{-1}{\rho-1}} + w_2^{\frac{\rho}{\rho-1}} \alpha_2^{\frac{-1}{\rho-1}} \right)\end{equation}\tag{12}\]The Taylor expansion of \(\frac{\rho-1}{\rho} \ln \left(w_1^{\frac{\rho}{\rho-1}} \alpha_1^{\frac{-1}{\rho-1}} + w_2^{\frac{\rho}{\rho-1}} \alpha_2^{\frac{-1}{\rho-1}} \right)\) around \(\rho=0\) is 22This is computed in Mathematica, too.\[\begin{align}\label{eq:taylor-expansion-of-log-cost}&((\alpha_2-1)\ln\alpha_1-\alpha_2(\ln\alpha_2+\ln w_1-\ln w_2)+\ln w_1)onumber\\&+\frac{1}{2}(\alpha_2-1)\alpha_2\rho(\ln \alpha_1-\ln \alpha_2-\ln w_1+\ln w_2)^2+O\left(\rho^2\right)\end{align}\tag{13}\]Omitting \(O\left(\rho ^2\right)\) and substituting the Taylor expansion into Equation Equation12, we have\[\begin{align}\ln C &= -\ln A + \ln Q onumber\\&+((\alpha_2-1)\ln\alpha_1-\alpha_2(\ln\alpha_2+\ln w_1-\ln w_2)+\ln w_1)onumber\\&+\frac{1}{2}(\alpha_2-1)\alpha_2\rho(\ln \alpha_1-\ln \alpha_2-\ln w_1+\ln w_2)^2\label{eq:log-cost-taylor}\end{align}\tag{14}\]which clearly is a function of \(\ln Q\); \(\ln w_1\), \(\ln w_2\) and their interaction terms.We can therefore reparameterize Equation Equation14 and get the Translog cost function:\[\begin{align} \label{eq:translog-cost}\ln C &= a_0 + a_1 \ln Q \\&+ b_{11} \ln w_1 + b_{22} \ln w_2 + b_{12} \ln w_1 \ln w_2 onumber\end{align}\tag{15}\]Why there is no interaction between \(\ln Q\) and \(\ln w\)?This is NOT an error! It is because we started from a standard CES production function, which doesnt include interaction terms.A more general form of translog cost function includes interaction terms \(\ln Q \ln w\) because the underlying production function is even more flexible than the standard CES production function. This is the beauty of translog.In a general form, the translog cost function \(\ln C(Q, W)\) as a function of output \(Q\) and a vector of \(n\) input prices \(W\) is represented as\[\begin{align}\ln C(Q, W) &= \beta_0 + \beta_1 \ln Q + \frac{1}{2} \beta_2 (\ln Q)^2 \\&+ \sum_{i=1}^{n} \gamma_i \ln W_i + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \theta_{ij} \ln W_i \ln W_j onumber \\&+ \sum_{i=1}^{n} \phi_i \ln Q \ln W_i onumber\label{eq:translog-cost-general}\end{align}\tag{16}\]Note that here it includes a quadratic term for \(\ln Q\) and interactions between \(\ln Q\) and \(\ln W\). As a result, it can approximate a wide range of very complex cost functions (hence complex underlying production function, via duality). In economic theory, a cost function is often assumed to be linearly homogeneous in input prices. This means that if all input prices \(W_i\) are scaled by a constant \(\lambda > 0\), the total cost \(C\) should also scale by the same constant \(\lambda\). Mathematically, this is expressed as:\[\begin{equation}C(Q, \lambda W) = \lambda C(Q, W)\end{equation}\]Linear homogeneity is an important property because it ensures that the cost function is consistent with the idea of constant returns to scale in prices. If we take the total differential of the log cost, holding output constant, we have,\[\begin{equation}d\ln C = \sum_{i=1}^{n} \gamma_i d\ln W_i+ \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \theta_{ij} \ln W_j d\ln W_i + \sum_{i=1}^{n}\phi_i \ln Q d\ln W_i\end{equation}\]By assumption, all input prices scale by the same factor \(\lambda\) so that \(d\ln W_i\) is the same across all \(n\) inputs. Therefore, we can factor it out, which gives,\[\begin{equation}d\ln C = d\ln \bar{W} \sum_{i=1}^{n} \gamma_i + d\ln \bar{W}^2 \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \theta_{ij} + d\ln \bar{W} \ln Q \sum_{i=1}^{n}\phi_i\end{equation}\]To ensure \(\frac{d\ln C}{d\ln \bar{W}}=1\) hence linear homogeneity in the translog cost function, the following conditions must be met:\[\begin{align}\sum_{i=1}^{n} \gamma_i &= 1 \\\sum_{j=1}^{n} \theta_{ij} &= 0 \quad \text{for all } i \\\sum_{i=1}^{n} \phi_{i} &= 0\end{align}\] Back to top

Ces production functions. Ces function cost function. Ces function. Ces production function cost minimization. Ces fee.