Related rates formulas

Related RatesIn this section, we will learn how to solve problems about related rates - these are questions in which there are two or more related variables that are both changing with respect to time. There are many different applications of this, so I'll walk you through several different types. They can all be solved by following several general steps:Identify all quantities given and to be determined, then make a sketch with these labeled.Write an equation which involves both the given and to be determined variablesUse Implicit Differentiation to differentiate the equation with respect to time (t).Plug the given variables into the differentiated equation and solve for the one which need to be determined It might be a good idea to go back over Implicit Differentiation for a review, as it is used to solve Related Rates.Example 1 - Ripples in a Pool A rock is dropped into a pool of water, causing ripples to form in expanding outward circles. The radius r of a ripple is increasing at a rate of 1 foot per second. When the radius is 6 feet, at what rate is the area A of the water inside the ripple changing? As a note, remember that a derivative is a rate of change. If we have a variable, the derivative of that variable is the rate that it is changing. For example, in this problem, we have the variable r; r is the radius of the ripple. dr/dt is the rate at which the ripple is changing - in this example, it is increasing at 1 foot per second. The same with A; A is the area, while dA/dt is the rate at which the area is changing. To do this problem, we first need to identify our variables - r is given, dr/dt is given, and A is to be determined. Here is a picture of this scenario: Now that we have identified our variables and drawn a sketch, let's find an equation that involves both r and A of a circle. Since r is a variable, dr/dt will be included once the equation is differentiated. These variables can be related by the equation for the area of a circle, A = π r2Differentiation with respect to t will obtain the related rate equation that we need to plug our information into: When the radius is 6 feet, the area is changing at a rate of 12π ft2/second, which is about 37.7 ft2/secondExample 2 - Ripples in a Pool Yes, we're going to do another problem with ripples in a pool; however, this time we will be solving for something other than the rate at which the area is expanding. This is to show how we can start with the same formula, but solve for a different rate of change. A rock is dropped into a pool of water, creating ripples which move outward from it. One of these ripples creates a circle with an area increasing at a rate of 30 ft2/sec. When the area is 25π square feet, at what rate is the radius expanding? As before, we need to make a sketch of the problem, listing and labeling our knowns and unknowns: Now, we need an equation that relates A and r of a circle - good thing we already know that from the last problem: A = π r2! Even though we've already done it, let's differentiate it again, just for practice: This was fine for the last problem, but notice that we are missing a variable; we have A and dA/dt, but we're missing r. Fortunately, we have a formula which relates the two right in front of us; since we know the A at the specific time that we need, we can use it to solve for r, using A. Using A = 25π, the radius is 5. Now we have all the information we need to solve for dr/dt! When the area is 25π ft2, the radius is expanding at 3/π, which is about .95, ft/s.Example 3 - An Inflating Balloon A spherical balloon is being inflated at a rate of 10 cubic feet per minute. Find how fast the radius is expanding when the radius is 1 foot. We need to name our knowns and unknowns, and label them on a drawing: Now we need an equation which relates A and r for a sphere; the volume of a sphere is Differentiating this equation, we get: Finally, we can solve for dr/dt and plug in our variables: When the radius is 1 foot, it is expanding at a rate of approximately .78 feet per minute.Example 4 - An Airplane Overhead An observer sees an airplane at a height of 5 miles moving toward him. If the distance s between him and the airplane is decreasing at a rate of 300 miles per hour when s is 12 miles, what is the speed of the airplane? With a problem such as this, a drawing is usually given, since the wording can be a bit confusing. Still, we need to list our knowns and unknowns, and also assign some variables of our own: Remember that since x in the above image is the horizontal distance, dx/dt is the rate at which the distance is changing, or the speed. Since we need the horizontal speed, dx/dt, a right triangle gives us the variable x to work with. First, let's use the Pythagorean Theorem to solve for x: Now we have all of our variables, and we also have an equation which relates s and x, so let's differentiate it and solve for the plane's speed, dx/dt: We got a negative number for the velocity; this is just because of the direction the airplane is moving, based on how we defined our variables in the beginning. Since we are only finding the speed, we don't need to worry about the negative sign; when s is 12 miles, and s is decreasing at 300 miles per hour, the speed of the airplane is 330.3 miles per hour.Example 6 - A Changing Angle of Elevation A camera is on the ground, filming a rocket launch. The rocket is rising according to the position function s = 30t2, where y is measured in feet and t in seconds. Find the rate of the change of the camera's angle at 15 seconds after the rocket initially launches. The camera is 1000 feet away from the rocket. First, let's list our variables and label them on a drawing; as with the last example, we'll need to use a right triangle to help us solve it: We need to solve for s. Since s is the height of the rocket, we can use the position function that we were given for it, and evaluate at 15 seconds: We also need ds/dt; this is the derivative of s, which is 60t. Now, using this information, we need to find the change in the angle of elevation of the camera when t = 15 and s = 6750. We can relate these using the trigonometric equation in our picture above: We are almost done, however we need to use our right triangle, along with some new terms, to find an expression for the cosine of our angle: Finally, we can substitute cosine for our new expression, and evaluate the problem: The angle of the camera at time t = 15 seconds is changing at approximately .02 radians per second. This article is written like a manual or guidebook. Please help rewrite this article from a descriptive, neutral point of view, and remove advice or instruction. (October 2015) (Learn how and when to remove this template message) Part of a series of articles aboutCalculus Fundamental theorem Leibniz integral rule Limits of functions Continuity Mean value theorem Rolle's theorem Differential Definitions Derivative (generalizations) Differential infinitesimal of a function total Concepts Differentiation notation Second derivative Implicit differentiation Logarithmic differentiation Related rates Taylor's theorem Rules and identities Sum Product Chain Power Quotient L'Hôpital's rule Inverse General Leibniz Faà di Bruno's formula Integral Lists of integrals Integral transform Definitions Antiderivative Integral (improper) Riemann integral Lebesgue integration Contour integration Integral of inverse functions Integration by Parts Discs Cylindrical shells Substitution (trigonometric, Weierstrass, Euler) Euler's formula Partial fractions Changing order Reduction formulae Differentiating under the integral sign Risch algorithm Series Geometric (arithmetico-geometric) Harmonic Alternating Power Binomial Taylor Convergence tests Summand limit (term test) Ratio Root Integral Direct comparison Limit comparison Alternating series Cauchy condensation Dirichlet Abel Vector Gradient Divergence Curl Laplacian Directional derivative Identities Theorems Gradient Green's Stokes' Divergence generalized Stokes Multivariable Formalisms Matrix Tensor Exterior Geometric Definitions Partial derivative Multiple integral Line integral Surface integral Volume integral Jacobian Hessian Specialized Fractional Malliavin Stochastic Variations Miscellaneous Precalculus History Glossary List of topics Integration Bee Analysis vte In differential calculus, related rates problems involve finding a rate at which a quantity changes by relating that quantity to other quantities whose rates of change are known. The rate of change is usually with respect to time. Because science and engineering often relate quantities to each other, the methods of related rates have broad applications in these fields. Differentiation with respect to time or one of the other variables requires application of the chain rule,[1] since most problems involve several variables. Fundamentally, if a function F {\displaystyle F} is defined such that F = f (x) {\displaystyle F=f(x)} , then the derivative of the function F {\displaystyle F} can be taken with respect to another variable. We assume x {\displaystyle x} is a function of t {\displaystyle t} , i.e. x = g (t) {\displaystyle x=g(t)} . Then F = f (g (t)) {\displaystyle F=f(g(t))} , so F ′ = f ′ (g (t)) ⋅ g ′ (t) {\displaystyle F'=f'(g(t))\cdot g'(t)} Written in Leibniz notation, this is: d F d t = d f d x ⋅ d x d t . {\displaystyle {\frac {dF}{dt}}={\frac {df}{dx}}\cdot {\frac {dx}{dt}}.} Thus, if it is known how x {\displaystyle x} changes with respect to t {\displaystyle t} , then we can determine how F {\displaystyle F} changes with respect to t {\displaystyle t} and vice versa. We can extend this application of the chain rule with the sum, difference, product and quotient rules of calculus, etc. For example, if F (x) = G (y) + H (z) {\displaystyle F(x)=G(y)+H(z)} then d F d x ⋅ d x d t = d G d y ⋅ d y d t + d H d z ⋅ d z d t . {\displaystyle {\frac {dF}{dx}}\cdot {\frac {dx}{dt}}={\frac {dG}{dy}}\cdot {\frac {dy}{dt}}+{\frac {dH}{dz}}\cdot {\frac {dz}{dt}}.} Procedure The most common way to approach related rates problems is the following:[2] Identify the known variables, including rates of change and the rate of change that is to be found. (Drawing a picture or representation of the problem can help to keep everything in order) Construct an equation relating the quantities whose rates of change are known to the quantity whose rate of change is to be found. Differentiate both sides of the equation with respect to time (or other rate of change). Often, the chain rule is employed at this step. Substitute the known rates of change and the known quantities into the equation. Solve for the wanted rate of change. Errors in this procedure are often caused by plugging in the known values for the variables before (rather than after) finding the derivative with respect to time. Doing so will yield an incorrect result, since if those values are substituted for the variables before differentiation, those variables will become constants; and when the equation is differentiated, zeroes appear in places of all variables for which the values were plugged in. Examples Leaning ladder example A 10-meter ladder is leaning against the wall of a building, and the base of the ladder is sliding away from the building at a rate of 3 meters per second. How fast is the top of the ladder sliding down the wall when the base of the ladder is 6 meters from the wall? The distance between the base of the ladder and the wall, x, and the height of the ladder on the wall, y, represent the sides of a right triangle with the ladder as the hypotenuse, h. The objective is to find dy/dt, the rate of change of y with respect to time, t, when h, x and dx/dt, the rate of change of x, are known. Step 1: x = 6 {\displaystyle x=6} h = 10 {\displaystyle h=10} d x d t = 3 {\displaystyle {\frac {dx}{dt}}=3} d h d t = 0 {\displaystyle {\frac {dh}{dt}}=0} d y d t = ? {\displaystyle {\frac {dy}{dt}}={\text{?}}} Step 2: From the Pythagorean theorem, the equation x 2 + y 2 = h 2 , {\displaystyle x^{2}+y^{2}=h^{2},\,} describes the relationship between x, y and h, for a right triangle. Differentiating both sides of this equation with respect to time, t, yields d d t (x 2 + y 2) = d d t (h 2) {\displaystyle {\frac {d}{dt}}(x^{2}+y^{2})={\frac {d}{dt}}(h^{2})} Step 3: When solved for the wanted rate of change, dy/dt, gives us d d t (x 2) + d d t (y 2) = d d t (h 2) {\displaystyle {\frac {d}{dt}}(x^{2})+{\frac {d}{dt}}(y^{2})={\frac {d}{dt}}(h^{2})} (2 x) d x d t + (2 y) d y d t = (2 h) d h d t {\displaystyle (2x){\frac {dx}{dt}}+(2y){\frac {dy}{dt}}=(2h){\frac {dh}{dt}}} x d x d t + y d y d t = h d h d t {\displaystyle x{\frac {dx}{dt}}+y{\frac {dy}{dt}}=h{\frac {dh}{dt}}} d y d t = h d h d t − x d x d t y . {\displaystyle {\frac {dy}{dt}}={\frac {h{\frac {dh}{dt}}-x{\frac {dx}{dt}}}{y}}.} Step 4 & 5: Using the variables from step 1 gives us: d y d t = h d h d t − x d x d t y . {\displaystyle {\frac {dy}{dt}}={\frac {h{\frac {dh}{dt}}-x{\frac {dx}{dt}}}{y}}.} d y d t = 10 × 0 − 6 × 3 y = − 18 y . {\displaystyle {\frac {dy}{dt}}={\frac {10\times 0-6\times 3}{y}}=-{\frac {18}{y}}.} Solving for y using the Pythagorean Theorem gives: x 2 + y 2 = h 2 {\displaystyle x^{2}+y^{2}=h^{2}} 6 2 + y 2 = 10 2 {\displaystyle 6^{2}+y^{2}=10^{2}} y = 8 {\displaystyle y=8} Plugging in 8 for the equation: − 18 y = − 18 8 = − 9 4 {\displaystyle -{\frac {18}{y}}=-{\frac {18}{8}}=-{\frac {9}{4}}} It is generally assumed that negative values represent the downward direction. In doing such, the top of the ladder is sliding down the wall at a rate of 9⁄4 meters per second. Physics examples Because one physical quantity often depends on another, which, in turn depends on others, such as time, related-rates methods have broad applications in Physics. This section presents an example of related rates kinematics and electromagnetic induction. Physics example I: relative kinematics of two vehicles One vehicle is headed North and currently located at (0,3); the other vehicle is headed West and currently located at (4,0). The chain rule can be used to find whether they are getting closer or further apart. For example, one can consider the kinematics problem where one vehicle is heading West toward an intersection at 80 miles per hour while another is heading North away from the intersection at 60 miles per hour. One can ask whether the vehicles are getting closer or further apart and at what rate at the moment when the North bound vehicle is 3 miles North of the intersection and the West bound vehicle is 4 miles East of the intersection. Big idea: use chain rule to compute rate of change of distance between two vehicles. Plan: Choose coordinate system Identify variables Draw picture Big idea: use chain rule to compute rate of change of distance between two vehicles Express c in terms of x and y via Pythagorean theorem Express dc/dt using chain rule in terms of dx/dt and dy/dt Substitute in x, y, dx/dt, dy/dt Simplify. Choose coordinate system: Let the y-axis point North and the x-axis point East. Identify variables: Define y(t) to be the distance of the vehicle heading North from the origin and x(t) to be the distance of the vehicle heading West from the origin. Express c in terms of x and y via the Pythagorean theorem: c = (x 2 + y 2) 1 / 2 {\displaystyle c=(x^{2}+y^{2})^{1/2}} Express dc/dt using chain rule in terms of dx/dt and dy/dt: d c d t = d d t (x 2 + y 2) 1 / 2 {\displaystyle {\frac {dc}{dt}}={\frac {d}{dt}}(x^{2}+y^{2})^{1/2}} Apply derivative operator to entire function = 1 2 (x 2 + y 2) − 1 / 2 d d t (x 2 + y 2) {\displaystyle ={\frac {1}{2}}(x^{2}+y^{2})^{-1/2}{\frac {d}{dt}}(x^{2}+y^{2})} Square root is outside function; Sum of squares is inside function = 1 2 (x 2 + y 2) − 1 / 2 [d d t (x 2) + d d t (y 2)] {\displaystyle ={\frac {1}{2}}(x^{2}+y^{2})^{-1/2}\left[{\frac {d}{dt}}(x^{2})+{\frac {d}{dt}}(y^{2})\right]} Distribute differentiation operator = 1 2 (x 2 + y 2) − 1 / 2 [2 x d x d t + 2 y d y d t] {\displaystyle ={\frac {1}{2}}(x^{2}+y^{2})^{-1/2}\left[2x{\frac {dx}{dt}}+2y{\frac {dy}{dt}}\right]} Apply chain rule to x(t) and y(t)} = x d x d t + y d y d t x 2 + y 2 {\displaystyle ={\frac {x{\frac {dx}{dt}}+y{\frac {dy}{dt}}}{\sqrt {x^{2}+y^{2}}}}} Simplify. Substitute in x = 4 mi, y = 3 mi, dx/dt = −80 mi/hr, dy/dt = 60 mi/hr and simplify d c d t = 4 mi ⋅ (− 80 mi / hr) + 3 mi ⋅ (60) mi / hr (4 mi) 2 + (3 mi) 2 = − 320 mi 2 / hr + 180 mi 2 / hr 5 mi = − 140 mi 2 / hr 5 mi = − 28 mi / hr {\displaystyle {\begin{aligned}{\frac {dc}{dt}}&={\frac {4{\text{ mi}}\cdot (-80{\text{ mi}}/{\text{hr}})+3{\text{ mi}}\cdot (60){\text{mi}}/{\text{hr}}}{\sqrt {(4{\text{ mi}})^{2}+(3{\text{ mi}})^{2}}}}\\&={\frac {-320{\text{ mi}}^{2}/{\text{hr}}+180{\text{ mi}}^{2}/{\text{hr}}}{5{\text{ mi}}}}\\&={\frac {-140{\text{ mi}}^{2}/{\text{hr}}}{5{\text{ mi}}}}\\&=-28{\text{ mi}}/{\text{hr}}\end{aligned}}} Consequently, the two vehicles are getting closer together at a rate of 28 mi/hr. Physics Example II: Electromagnetic induction of conducting loop spinning in magnetic field The magnetic flux through a loop of area A whose normal is at an angle θ to a magnetic field of strength B is Φ B = B A cos (θ) , {\displaystyle \Phi _{B}=BA\cos(\theta),} Faraday's law of electromagnetic induction states that the induced electromotive force E {\displaystyle {\mathcal {E}}} is the negative rate of change of magnetic flux Φ B {\displaystyle \Phi _{B}} through a conducting loop. E = − d Φ B d t , {\displaystyle {\mathcal {E}}=-{{d\Phi _{B}} \over dt},} If the loop area A and magnetic field B are held constant, but the loop is rotated so that the angle θ is a known function of time, the rate of change of θ can be related to the rate of change of Φ B {\displaystyle \Phi _{B}} (and therefore the electromotive force) by taking the time derivative of the flux relation E = − d Φ B d t = B A sin θ d θ d t {\displaystyle {\mathcal {E}}=-{\frac {d\Phi _{B}}{dt}}=BA\sin \theta {\frac {d\theta }{dt}}} If for example, the loop is rotating at a constant angular velocity ω, so that θ = ωt, then E = ω B A sin ω t {\displaystyle {\mathcal {E}}=\omega BA\sin \omega t} References ^ "Related Rates". Whitman College. Retrieved 2013-10-27. ^ Kreider, Donald. "Related Rates". Dartmouth. Retrieved 2013-10-27. Retrieved from "